Journal of Engineering Physics and Thermophysics, Vol. 80, No. 5, 2007

PROPAGATION OF STRAIN WAVES AND COMPACTION OF A VISCOELASTIC MATERIAL BY A CYLINDER ROLLING OVER IT

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UDC 539.379.24:631.431.73:001.573

This paper considers problems on the rolling of rigid driven and driving cylinders slipping on a compactible viscoelastic base. As a result of the investigation of the process of propagation of viscoelastic decaying strain waves in a deformable medium caused by a cylinder rolling over it and the action of the sliding friction forces, formulas and calculation algorithms have been obtained for determining the indices of the stressed-strained state and compactness of the base, the rolling resistance of the driven and diving cylinders, and the driving force of the driving cylinder. Calculations of these indices have been performed for the cases of cylinders rolling over a soil whose viscoelastic properties have been investigated experimentally.

The problem of rolling friction is very important in mechanics. Many investigations have been carried out in this area [1–12]. However, this problem has not been solved completely. In [5–7], a review of some of the results obtained is given. The problems on the rolling of a rigid, an elastic, and a viscoelastic cylinder over a plastic, an elastic, and a viscoelastic base have been considered. In [2–7], as in a number of other works, the contact arc was taken to be small compared to the cylinder radius, and the boundary conditions were given on a segment of a straight line.

This paper presents the results of an investigation of the rolling of circular rigid cylinders over a compactible disperse base. We took into account two main factors of the appearance of the stressed-strained state of the base and the rolling resistance — the viscoelastic properties of the deformable medium and the sliding friction on the contact surface of the cylinder and base. Unlike [2–7], the contact line is considered by us as a circular arc (i.e., the actually existing one), and is not replaced approximately by a segment of a straight line or a parabolic arc. The boundary conditions are given on a circular arc.

Let us consider the rolling with negative or positive sliding (hereinafter referred to as sliding or slippage, respectively) of a rigid circular cylinder of radius R over a horizontal surface of a dispersive medium. The velocity $\mathbf{v}_0$ of the cylinder axis and its angular velocity ω are assumed to be constant. The slippage coefficient of the cylinder $\delta = 1 - v_0/\omega R$. The deformable layer of the medium is extended to depth H.

Let us introduce a stationary and a rectangular coordinate systems Oxyz and O$_1x_1y_1z_1$. The origin O of the stationary coordinate system coincides at the initial instant of time t (at $t = 0$) with the point A at which the cylinder comes into contact with the base. The direction of the horizontal axis Ox passing over the base surface coincides with the direction of motion of the cylinder axis; the Oy-axis is directed vertically downward; the Oz-axis is parallel to the cylinder axis (perpendicular to the plane) (Fig. 1a). The velocity $\mathbf{v}_0$ is codirected with Ox. The coordinate system O$_1x_1y_1z_1$ with the origin at the point O$_1$ located on the cylinder axis moves together with its axis with velocity $\mathbf{v}_0$: O$_1x_1 \| Ox$, O$_1y_1 \| Oy$, O$_1z_1 \| Oz$. Let us also introduce a cylindrical coordinate system R, ψ, z_1 with a pole at the point O$_1$ whose polar axis coincides with the O$_1y_1$-axis, and the O$_1z_1$-axis coincides with the corresponding axis of the rectangular coordinate system. The polar radius of any point located on the cylinder-base contact line is equal to R. At $x_1 \geq 0$ the polar angle $\psi \geq 0$, and at $x_1 < 0$ the angle $\psi < 0$.

Let the density of the deformable medium be ρ = const at all $y \leq H$. The cylinder length L is taken to be large enough; therefore, the deformation of the base is approximately plane. The cylinder-base contact line in the longitudinal section of the cylinder is a circular arc BCA.

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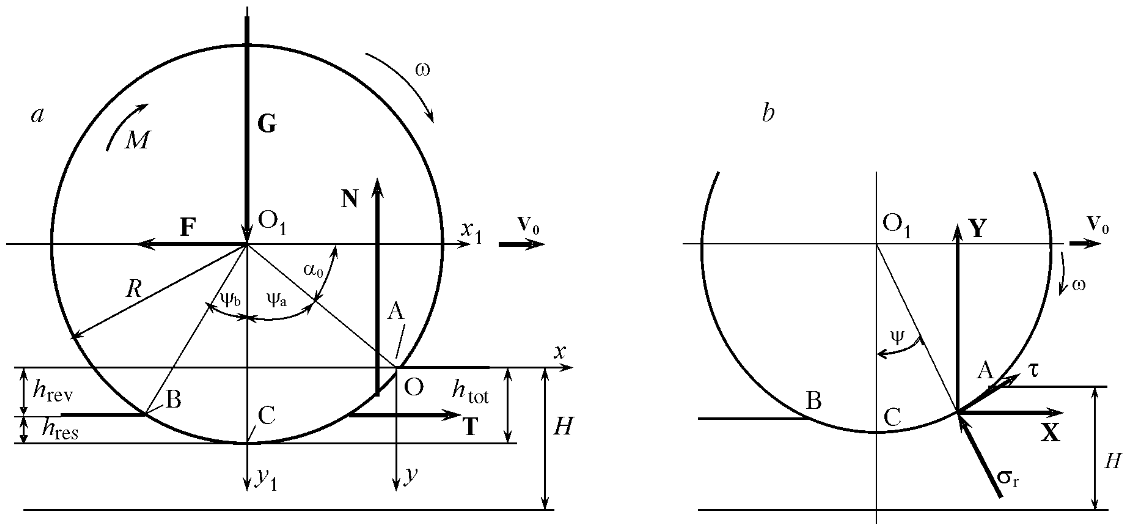


Fig. 1. Scheme of the interaction between the driving circular cylinder and the deformable base (a) and components of the elementary reactions of the base distributed along the contact line BA (b) (A and B are the points at which the cylinder comes into contact with the base and goes out of this contact; BCA is the cylinder-base contact line (circular arc of radius R); C is a point of the BCA line situated on the Oy_1-axis).

The cylinder is subjected to the action of the forces applied to it — the vertical force **G** and the horizontal force **F**, the torque M, and the reactions of the base. The reactions distributed over the contact surface are replaced by the resultants — the vertical force **N** and the horizontal force **T**. The values of **N** and **T** depend on the viscoelastic properties of the base and the sliding friction forces on the contact surface. In the driving cylinder, the directions of M and ω coincide, $\delta > 0$, and the reaction **T** is directed towards the motion of the axis. For this case, we took $T > 0$ and the value of the driving torque $M > 0$. In the driven cylinder, M and ω are opposite, $\delta < 0$, $T < 0$, and the drag torque $M < 0$.

The elementary reactions of the base (contact stresses) distributed along the contact line have at each point of this line radial (normal) $\boldsymbol{\sigma}_r$ and its tangential $\boldsymbol{\tau}$ components (Fig. 1b). The values of the horizontal and vertical components of the contact stress are

$$X = -\sigma_x \pm \tau_x\,, \quad Y = \sigma_y \pm \tau_y\,, \tag{1}$$

where $\tau_x = \tau \cos \psi$; $\tau_y = \tau \sin \psi$; $\sigma_x = \sigma_y \tan \psi$. The value of $\boldsymbol{\tau}$ depends on σ_r, δ, and f. For the two signs "$\pm$" in formula (1) and the following formulas given below, the signs "+" are used for the driving cylinder and "−" for the driven one.

On the contact line, the deformations of the base and the elementary reactions distributed along it are functions of one variable t or the current angle $\psi = \psi_a - \omega t$ ($\psi \in [\psi_b, \psi_a]$, $\psi_b < 0$, $\psi_a > 0$). At $\psi \in [0, \psi_a]$ compression of the viscoelastic medium occurs, and at $\psi \in [\psi_b, 0]$ its reversible deformation is observed; $\psi_a > |\psi_b|$. In the depth of the deformable medium layer (at $y > 0$), the stresses σ_y, the horizontal u and vertical v shifts of the medium with a cylinder rolling over it are functions of y and t: $\sigma_y = \sigma_y(y, t)$, $u = u(y, t)$, and $v = v(y, t)$.

The relations between the compressive deformation components and the compressive stresses in an isotropic viscoelastic medium can be described by the following differential equations:

$$\frac{\partial \varepsilon_y}{\partial t} = \frac{1+\mu}{E_d}\left[(1-\mu)\left(\frac{\partial \sigma_y}{\partial t} + \beta\sigma_y\right) - \mu\left(\frac{\partial \sigma_x}{\partial t} + \beta\sigma_x\right)\right],$$

$$\frac{\partial\varepsilon_x}{\partial t}=\frac{1+\mu}{E_{\rm d}}\left[(1-\mu)\left(\frac{\partial\sigma_x}{\partial t}+\beta\sigma_x\right)-\mu\left(\frac{\partial\sigma_y}{\partial t}+\beta\sigma_y\right)\right]. \tag{2}$$

If the relation between the time-variable compressive stresses and deformations is revealed on the basis of die tests performed with the possibility of lateral expansion on the medium, then for generalized mathematical modeling of this relation one equation can be used. For the rolling cylinder, the stresses σ_x at the contact line points are expressed in terms of σ_y; therefore, the first equation in system (2) can be replaced by a differential equation whose right-hand side includes only the component σ_y and its time derivative. Using the results of the investigations of [8–11, 13], we shall model the mechanism of viscoelastic medium compression in the direction of the y-axis by the differential equation

$$\frac{\partial\varepsilon_y(y,t)}{\partial t}=\frac{1}{q}\left(\frac{\partial\sigma_y(y,t)}{\partial t}+p\sigma_y(y,t)\right). \tag{3}$$

The parameters p and q of Eq. (3) are characteristics of the viscoelastic properties of the medium. They complexly describe the viscoelastic properties of the medium without dividing it into an elastic and a viscous components and take into account its lateral expansion under the action of a load. In the case of deformation by the harmonic mechanism, $p = \omega g$, where ω is the frequency of this deformation process. The characteristics $g = p/\omega$ and q depend on the density ρ of the medium, its humidity w, and the frequency ω.

The compressive deformation of the base propagates to a depth $H_{\rm p} \le H$. The finite relative compressive deformation of the base at the contact line points is

$$\varepsilon_y(0,\psi)=h(0,\psi)/H_{\rm p}=R(\cos\psi-\cos\psi_{\rm a})/H_{\rm p}, \tag{4}$$

where $h(0, \psi) = R(\cos\psi - \cos\psi_{\rm a})$ is the absolute compressive deformation (settlement) of the base. Using (3), (4) and the boundary conditions $\sigma_y(0, \psi_{\rm a}) = 0$ and $\sigma_y(0, \psi_{\rm b}) = 0$, we obtain the formula for determining the compressive stresses of the medium at the contact line points

$$\sigma_y(0,\psi)=\frac{qR}{H_{\rm p}(g^2+1)}\left[\cos\psi+g\sin\psi-(\cos\psi_{\rm a}+g\sin\psi_{\rm a})\exp(-g(\psi_{\rm a}-\psi))\right] \tag{5}$$

and the equation relating $\psi_{\rm a}$ and $\psi_{\rm b}$:

$$\exp(-g\,\psi_{\rm a})(\cos\psi_{\rm a}+g\sin\psi_{\rm a})-\exp(-g\,\psi_{\rm b})(\cos\psi_{\rm b}+g\sin\psi_{\rm b})=0. \tag{6}$$

In the deformation of a viscoelastic medium, a shift of the phases of deformations and stresses is observed: the points of their maxima do not coincide. For the rolling cylinder, the maximum of the deformation $(\varepsilon_y)_{\rm m}$ corresponding to the points on the contact line is achieved at $\psi = 0$. We determine the angle $\psi_{\rm m}$ at which σ_y has the maximum value $(\sigma_y)_{\rm m}$ as a solution of the equation obtained from the condition $(\partial\sigma_y(0, \psi)/\partial\psi\,|_{\psi=\psi_{\rm m}} = 0$. The stress $(\sigma_y)_{\rm m}$ is determined by formula (5) at $\psi = \psi_{\rm m}$.

On the contact surface of the cylinder, because of the action of the friction forces, zones of adhesion and relative sliding are formed [4–12]. However, in [2, 4, 8] and other works, the N value is determined without taking into account the influence of the friction forces between the cylinder and the base. In the present work the friction forces are included into the N value. In determining N, let us assume that the whole of the contact surface represents a relative sliding zone, i.e., at all points of the contact line $\tau_y = \pm f\sigma_y \tan\psi$. This conforms to the real cases of cylinder rolling at certain $\psi_{\rm a}$, $\psi_{\rm b}$, f, and δ. The value of the resultant of the vertical reaction of the base is

$$N=LR\int_{\psi_{\rm b}}^{\psi_{\rm a}}(1\pm f\tan\psi)\,\sigma_y(0,\psi)\,d\psi. \tag{7}$$

If besides the sliding zone on the contact surface there are also adhesion zones (in these zones $|\tau_y| < |f\sigma_y \tan\psi|$), then N is determined from (7) with a high accuracy ($\tau_y << \sigma_y$ and the substitution of τ_y by a somewhat larger value on a part of the contact surface has no marked effect on the calculation data).

From the conditions of steady motion of the cylinder it follows that $N = G$. Taking into account this equality, by manipulations in (7) we get

$$G - LR^2 q\,[(\cos\psi_b - \cos\psi_a)/g \pm f\,((\cos\psi_b - \cos\psi_a) + g\,(\cos\psi_b - \sin\psi_a) + g\ln\left|\frac{\tan((\pi/4)+\psi_a/2)}{\tan((\pi/4)+\psi_b/2)}\right| -$$

$$- (\cos\psi_b + g\sin\psi_b)\exp(-g\,\psi_b)\int_{\psi_b}^{\psi_a}\exp(g\,\psi)\tan\psi d\psi)/(g^2+1)]/H_p = 0\,. \tag{8}$$

The integral entering into (8) has been calculated approximately.

For the cylinder rolling in the plane perpendicular to its axis and passing through the middle of the axis, a planar deformation wave propagates. It consists of a compression wave of the viscoelastic medium caused by the vertical shifts of the medium and a shear wave caused by its horizontal shifts. The medium compression under the action of the stress σ_y is described by the differential equation of motion

$$\frac{\partial\sigma_y}{\partial y} = \rho\,\frac{\partial^2 v}{\partial t^2}\,. \tag{9}$$

Taking into account that $\varepsilon_y = \partial v/\partial y$, as a result of using (3) and (9), we obtain a differential equation with third-order partial derivatives which models the compression wave propagation in the viscoelastic medium at ρ = const:

$$\rho\,\frac{\partial^3 v}{\partial t^3} - q\,\frac{\partial^3 v}{\partial y^2\partial t} + p\rho\,\frac{\partial^2 v}{\partial t^2} = 0\,. \tag{10}$$

Let us solve Eq. (10) that satisfies the conditions reflecting the physical picture of the interaction between the rolling cylinder and the deformable medium. We first assume that with a rolling cylinder $H_p < H$. This permits considering the region of deformation propagation to be unbounded below, i.e., $H \to \infty$. The cylinder-base interaction time in one rotation of the cylinder about its axis $t_{in} = (\psi_a + |\psi_b|)/\omega$. The problem in determining $v(y,\ t)$ at $t \in [0,\ t_{in}]$ is as follows: solve Eq. (10) satisfying the boundary conditions

$$\begin{cases} v\,(0,t) = R\,(\sin(\alpha_0 + \omega t) - \sin\alpha_0)\,, \\ v\,(\infty, t) = 0\,, \end{cases} \qquad t \in [0, t_{in}] \tag{11}$$

and the initial conditions

$$\begin{cases} v\,(y,0) = 0\,, \\ \partial v\,(y,0)/\partial t = 0\,, \end{cases} \qquad y \in [0,\infty)\,. \tag{12}$$

The solution of the problem (10)–(12) was sought on the basis of the results of [14] in the form

$$v\,(y,t) = \begin{cases} 0 & \text{at } t \in (-\infty, 0]\,, \\ R\exp(-f_0 y)\,[\sin(\alpha_0 + \omega t - c_0 y) - \sin\alpha_0] & \text{at } t \in (0, t_{in}]\,, \end{cases} \qquad y \cup [0,\infty)\,. \tag{13}$$

The coefficients $f_0 > 0$ and $c_0 > 0$ should be determined so that (13) is the solution of Eq. (10). In (13) at $t > 0$ the shifts $v(y,\ t) \geq 0$.

Let us substitute the expressions $\partial^2 v/\partial t^2$, $\partial^3 v/\partial t^3$, $\partial^3 v/\partial y^2 \partial t$ obtained from (13) into (10). By manipulations and setting the coefficients at $\cos(\alpha_0 + \omega t - c_0 y)$ and $\sin(\alpha_0 + \omega t - c_0 y)$ to zero, we get the system of equations

$$\rho\omega^2 + q\,(f_0^2 - c_0^2) = 0\,, \quad 2qf_0c_0 - p\rho\omega = 0\,, \tag{14}$$

from which we find the unknowns f_0 and c_0:

$$f_0 = \omega\sqrt{\rho\,(\sqrt{g^2+1} - 1)/2q}\,, \tag{15}$$

$$c_0 = \frac{\rho\omega^2 g}{2qf_0} = \omega g\sqrt{\frac{\rho}{2q\,(\sqrt{g^2+1} - 1)}}\,. \tag{16}$$

Expression (13) with the values of f_0 and c_0 found by formulas (15) and (16) satisfies the boundary conditions and the initial conditions.

We determine the propagation depth of compressive deformation of the medium $H_{\text{t.p}}$ which is theoretically possible at $H \to \infty$ from the condition $v(H_{\text{t.p}}, t_{\text{in}}) = 0$. Satisfying this condition and making use of expression (13), we get

$$\psi_{\text{a}} + |\psi_{\text{b}}| - c_0 H_{\text{t.p}} = 0\,. \tag{17}$$

In the formulas obtained the unknowns are ψ_{a}, ψ_{b}, and $H_{\text{t.p}}$. They are defined as a solution of the system of equations (6), (8), and (17) in three unknowns. In solving this system, the depth $H_{\text{t.p}}$ enters into (8). We have developed an algorithm for solving the system of equations (6), (8), (17) with any given accuracy in given domains of variability of the unknowns. In these domains, the system is definite. If upon solving this system $H_{\text{t.p}} \le H$ is obtained, then $H_{\text{p}} = H_{\text{t.p}}$. Using expressions (5), (9), and (13), we find the formula for determining the maximal compressive stresses attenuating with depth at $H_{\text{p}} \le H$:

$$\sigma_y(y, \psi_{\text{m}}) = \sigma_y(0, \psi_{\text{m}}) - \rho\omega^2 R\,[(c_0 \sin(\psi_{\text{m}} + c_0 y) -$$

$$- f_0 \cos(\psi_{\text{m}} + c_0 y))/\exp(f_0 y) + c_0 \cos\psi_{\text{m}} - f_0 \sin\psi_{\text{m}}]/(f_0^2 + c_0^2)\,. \tag{18}$$

If upon solving the system of equations (6), (8), (17) $H_{\text{t.p}} > H$ is obtained, then it means that the region of deformation propagation of the medium is bounded below by the value of H, with $H_{\text{p}} = H$. The boundary $y = H$ excites a reflected wave [15]. In this case, ψ_{a} and ψ_{b} are determined as a solution of a certain system of two transcendental equations (6) and (8) in two unknowns.

On the basis of the results presented in [15], in solving the boundary-value problem (10)–(12), we seek $H_{\text{p}} = H$ in the form of the series

$$v(y, t) = R\left(\sum_{n=0}^{\infty} \exp(-f_n y)\,(\sin(\alpha_0 + \omega t - c_n y) - \sin\alpha_0)\right) -$$

$$- R\left(\sum_{n=1}^{\infty} \exp(-l_n y)\,(\sin(\alpha_0 + \omega t - r_n y) - \sin\alpha_0)\right), \tag{19}$$

where f_n, c_n, l_n, r_n are coefficients which should be determined so that expression (19) is the solution of Eq. (10). Expression (19) with the found values of the above coefficients satisfies the boundary and initial conditions of the problem. Each sum in (19) contains a finite number of nonzero terms, since the condition $v(y, t) \ge 0$ should be satisfied.

Formula (19) has the following physical meaning. The term obtained at $n = 0$ describes the wave excited by the boundary conditions at $y = 0$ independent of the influence of the boundary $y = H$, as if the base was deformed to an infinitely large depth. The following terms (upon multiplication by R) describe the waves representing sequential reflections from the boundary $y = H$ (the second sum multiplied by R) and from the base surface $y = 0$ (the first sum multiplied by R). The coefficients f_n, c_n, l_n, r_n are determined sequentially at $n = 0, 1, 2, \ldots$ for each direct and reflected wave; f_0 and c_0 are calculated by formulas (15) and (16).

The viscoelastic properties of the deformable medium are reflected by the coefficient

$$k_{\text{rev}} = h_{\text{rev}}(0, t_{\text{in}})/h_{\text{tot}}(0, t_{\text{in}}) = (1 - \cos\psi_{\text{b}})/(1 - \cos\psi_{\text{a}}) . \tag{20}$$

Having determined ψ_{a}, ψ_{b}, and H_{p}, we find $h_{\text{res}} = R(\cos\psi_{\text{b}} - \cos\psi_{\text{b}})$, h_{rev}, and k_{rev}.

The result of the solution of the problem in determining $v(y, t)$ at $t \le t_{\text{in}}$ and the numerical values of ψ_{a} and ψ_{b}, and H_{p} have been used to form the initial conditions of the problem in determining $v(y, t)$ at $t > t_{\text{in}}$. It is necessary to solve Eq. (10) satisfying the boundary conditions

$$v(0, t) = h_{\text{res}}, \quad v(H_{\text{p}}, t) = 0 \quad (t \in (t_{\text{in}}, \infty)) \tag{21}$$

and the initial conditions

$$v(y, t_{\text{in}}) = \varphi_1(y), \quad \partial v(y, t_{\text{in}})/\partial t = \varphi_2(y), \quad \partial^2 v(y, t_{\text{in}})/\partial t^2 = \varphi_3(y) \quad (y \in [0, H_{\text{p}}]) . \tag{22}$$

We determine the functions $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$ from (13) at $t = t_{\text{in}}$. The problem (10), (21), (22) has been solved approximately with the use of the results of [15], the Laplace–Carson transform, and the collocation method of [16] in the form

$$v(y, t) = \frac{h_{\text{res}}(H_{\text{p}} - y)}{H_{\text{p}}} + \sum_{i=1}^{s} C_i(t) \sin\frac{\pi i}{H_{\text{p}}} y . \tag{23}$$

At $t \to \infty$ we get: $C_i(t) \to \tilde{C}_i$, $v(y, \infty) \to v_{\text{st}}(y)$. If the boundary $y = H$ excites a reflected wave, then $\tilde{C}_i \approx 0$ and $v_{\text{st}}(y)$ are determined approximately by the first term on the right-hand side of formula (23).

In the present paper, it is assumed that the density increment $\Delta\rho$ at the depth $y + v_{\text{st}}(y)$ is proportional to its stabilized shifts $v_{\text{st}}(y) = v(0, \infty)$:

$$\Delta\rho(y + v_{\text{st}}(y)) = K_{\text{comp}} v_{\text{st}}(y) . \tag{24}$$

We determine the coefficient K_{comp} proceeding from the condition: the mass of the deformable medium moving under the action of the rolling cylinder from the upper layer corresponding to $y \in [0, h_{\text{res}})$ is equal to the increment of the underlying layer mass corresponding to $y \in [h_{\text{res}}, H_{\text{p}}]$. If $s = 3$, then

$$K_{\text{comp}} = \frac{2\rho h_{\text{res}} H_{\text{p}}}{(H_{\text{p}} + \mu h_{\text{res}})^2 (h_{\text{res}} + 4(C_1 + C_3/3)/\pi)} . \tag{25}$$

At $y = 0$ we have $v_{\text{st}}(0) = h_{\text{res}}$, therefore

$$\Delta\rho(h_{\text{res}}) = K_{\text{comp}}(h_{\text{res}}) . \tag{26}$$

Upon a pass of the cylinder the propagation depth of the deformable layer of the base $\tilde{H} = H - h_{\text{res}}$. As a new reference point of the depth, i.e., the quantity $\tilde{y}$ (i.e., as a new surface of the deformable medium), we take the coordinate $y = h_{\text{res}}$. Knowing the density increment of the deformable medium at different depths upon a pass of the cylinder, we find the new values of its density. As a result of the solution of the problems on the cylinder rolling at $\delta > 0$ and $\delta < 0$, we also obtained, apart from formulas (4)–(6), (8), (15)–(19), (23)–(26), the relations for determining M, the values of the forces $\mathbf{S}$, $\mathbf{F} = \mathbf{T}$, etc.

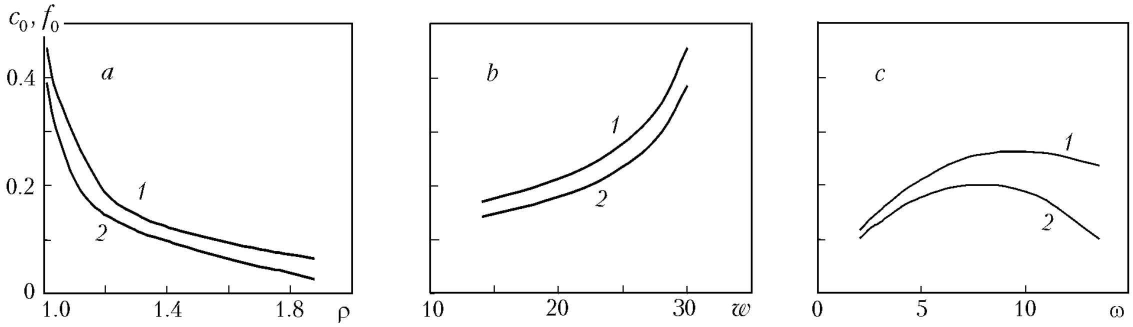


Fig. 2. Coefficients f_0 and c_0 as a function of the density of the soil (a), its humidity (b), and the angular rate of deformation (c): 1) f_0, 2) c_0 ($\rho = 1.118$, $\omega = 5.19$, $w = 20\%$); a) $\omega = 5.19$, $w = 20$; b) $\rho = 1.118$, $\omega = 5.19$; c) $\rho = 1.118$, $w = 20\%$. ρ, g/cm^3; w, %; ω, sec^{-1}.

In [2–5] and a number of other works, for modeling the rheological properties of the base that deforms under the action of the rolling cylinder, we used some of the constitutive equations of the viscoelasticity theory given in [6, 7, 17]. However, no experimental check of the applicability of the constitutive equations has been carried out for particular real deformable media.

We have made theoretical and experimental studies of the rolling with a slip of cylinders (rollers and wheels) on a soil [8–10]. At humidities lower than the total moisture capacity the soil is compacted and strengthened under the action of a load. At such w noncompacted soils are viscoelastic. On the basis of the analysis of the experimentally elucidated mechanisms of deformation of compatible soils it was suggested [8–10, 13] to model their rheological properties by the differential equation (3).

The limits within which Eq. (3) can be used for specific soils are determined experimentally. For instance, we have confirmed its suitability and advantages for sod-podzol slightly loamy soils of a certain mechanical composition at certain w, t, σ, ε [8, 10, 13], as well as for black earths [11]. For the investigated sod-podzol soil, we have derived (at $\rho = 1.138$–1.579 g/cm^3,$\omega = 16$–26%, and $\omega = 0.93$–5.01 sec^{-1}) the following regression equations:

$$g = 14.655 - 6.716\rho - 0.581\omega + 0.085w\,, \tag{27}$$

$$q = 14.981\rho + 0.245\omega - 0.315w - 9.654\,. \tag{28}$$

In the present work, we have investigated the influence on the coefficients f_0 and c_0 characterizing the attenuation with depth of the compression wave of this soil, its density ρ, humidity w, and frequency ω. It has been found that f_0 and c_0 markedly decrease as ρ increases from 1.1 to 1.9 g/cm^3 and increase appreciably as w increases from 14 to 30% and ω increases from 2.1 to 5 sec^{-1} (Fig. 2). If we assume that at $5.01 < \omega < 14.2$ sec^{-1} relations (27) and (28) hold, then the dependences of f_0 and c_0 on ω are characterized by curves having maxima (Fig. 2c).

As a result of the investigation performed, we have developed a method for calculating at $\sigma_y < \sigma_{str}$ the indices of the stressed-strained state and compaction of a viscoelastic dispersive medium and other indices for the rolling of the driven and driving cylinders and a computer program realizing it. The program permits calculating q, g, p, f_0, c_0, $H_{t.p}$, H_p, ψ_a, ψ_b, ψ_m, $(\sigma_y)_m$, k_{rev}, S, T, M, $h(0, \psi)$, $\varepsilon_y(0, \psi)$, $\sigma_y(0, \psi)$ at various $\psi \in [\psi_b, \psi_a]$, h_{res}, $\Delta\rho(y + v_{st}(y))$, and other indices. In the investigations of the rolling of cylinders over a soil, we took into account the changes in ρ and μ given at the soil humidity w_{pr} at its changed new value of its humidity w_{new}.

The input data for performing the calculations are: R, L, G, v_0, δ, ρ, μ, f, w_{pr}, w_{new}, H and the coefficients of the dependences $g(\rho, \omega, w)$, $q = q(\rho, \omega, w)$. With the aid of the programs developed, we performed in the present work calculations — computer experiments — in which we determined the indices under investigation. In the computer experiments, we considered the rolling with sliding of a roller as well as of the driving cylinder over the soil whose viscoelastic properties were investigated in [8–10, 13].

TABLE 1. Influencing Factors and Intervals of Their Chang

Influencing factors	Base calculation (in one-factor experiments)	Intervals of variation of factors in one-factor experiments	Lower/upper levels of variation of factors in complete factor experiments of the 2^3 type
G, kN	5	3–8	1/5
v_0, m/sec	2.1	1–5	2/4
w, %	20	14–26	14/24
δ	—0.1	–0.3–(+0.2)	—

TABLE 2. Indices of the Roller-Soil Interaction under Different Conditions

Indices	Influencing factors					
	G, kN		v_0, m/sec		w, %	
	3.0	8.0	1.0	5.0	14	26
$(\sigma_y)_m$, kPa	22.18	42.84	27.64	37.74	35.24	25.81
ψ_a, deg	19.67	31.48	33.52	14.26	20.85	33.71
ψ_b, deg	– 8.28	– 9.44	– 7.66	– 11.82	– 8.58	– 9.38
ψ_m, deg	8.63	16.86	20.15	1.82	9.31	18.66
h_{tot}, cm	2.04	5.15	5.82	1.08	2.29	5.88
h_{res}, cm	1.68	4.68	5.51	0.34	1.90	5.42
k_{rev}	0.18	0.09	0.05	0.69	0.17	0.08
$\Delta\rho(h_{res})$, g/cm^3	0.0376	0.1029	0.1205	0.0076	0.0413	0.1303
$\tilde{\rho}_0$, g/cm^3	1.1710	1.2363	1.2539	1.1410	1.1113	1.2247
$\tilde{\rho}(0.05)$, g/cm^3	1.1601	1.2311	1.2478	1.1406	1.1112	1.2156
S, kN	0.38	3.95	1.78	0.13	0.69	1.58
F, kN	1.79	6.68	4.96	2.92	3.0	3.49
t_{in}, sec	0.0894	0.1309	0.2767	0.0350	0.0942	0.1379

Note. Base calculation: G = 5 kN, v_0 = 2.1 m/sec, w = 20%, δ = –0.1.

We have carried out four sets of one-factor experiments (each set consisting of nine experiments) and a complete factor experiment of the type of 2^3, in which G, v_0, and w were varied. Table 1 gives the values of G, v_0, w, and δ for the basic calculation. The factors differing from those that were varied in the given series were assumed to be the same as in the basic calculation.

The input data for calculating the indices for a pass of the roller are: R = 0.35 m, L = 1.4 m, H = 1 m, w_{pr} = 18.37%, ρ = 1.118 g/cm^3, f = 0.32, μ = 0.318, δ = –0.1. The characteristics g and q of the soil were determined by Eqs. (27) and (28).

We have determined the indices of the roller-soil interaction under different conditions (Table 2). The hysteresis loops, the $\sigma_y(\varepsilon_y)$ curves plotted at y = 0, characterize the viscoelastic properties of the soil. An increase in G from 3 to 5 kN leads to an increase in σ_y and $(\varepsilon_y)_{res}$. As v_0 increases from 2 to 4 m/sec, the stresses σ_y and the deformations markedly decrease (Fig. 3).

In all experiments, it has been obtained that $H_{t.p} > H$; therefore, $H_p = H$ and $v(y, t)$ are described by formula (19). The calculations have shown that in the considered cases one direct wave was excited on the soil surface at y = 0 and one wave was reflected from the boundary $y = H$ arise. The reflected wave attenuates before it reaches the soil surface. We have determined the total shifts of the soil caused by the rolling of the cylinder with different velocities and $t = t_{in}$ as a shift difference $v(y, t_{in})$ of the propagating direct and reflected waves (Fig. 4).

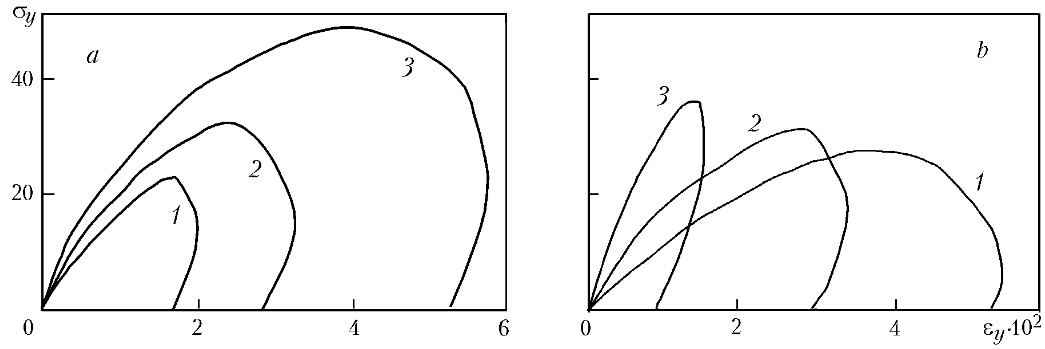
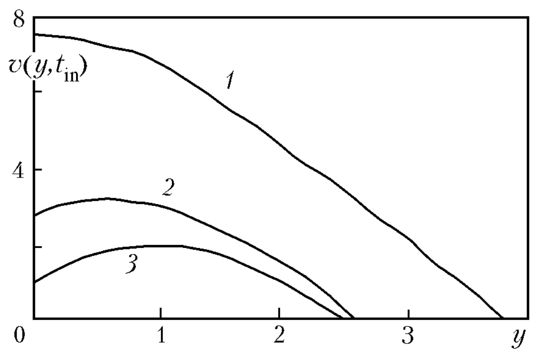


Fig. 3. Hysteresis loops for the cylinder rolling over the viscoelastic soil: a) at different vertical loadings on the axis [1, 2, and 3) G = 3, 5, and 9 kN (v_0 = 2 m/sec; w = 20%)]; b) at different velocities [1, 2, and 3) v_0 = 1, 2, and 4 m/sec (G = 5 kN, w = 20%)]. σ_y, kPa.



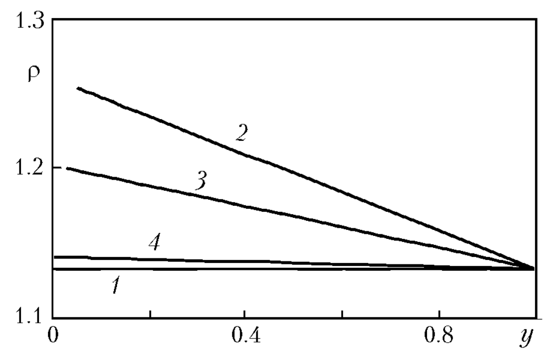


Fig. 4. Change in the vertical shifts of the soil with the depth of its deformable layer caused by the driving cylinder rolling at different velocities with a below-unbounded region of deformation propagation: 1, 2, and 3) v_0 = 2, 3, and 4 m/sec (G = 5 kN, δ = 0.1, $H \to \infty$). $v(y, t_{in})$, cm; y, m.

Fig. 5. Dependence of the soil density on the depth (G = 5 kN, w = 20%): 1) before the pass of the roller; 2, 3, and 4) at v_0 = 1, 2, and 5 m/sec. ρ, g/cm^3; y, m.

As a result of the calculations, it has been obtained that the soil density increment after a pass of the roller at depth h_{res} increases with increasing G and w and decreases with increasing v_0 (Table 2). The dependence of the soil density on the depth after a pass of the roller is described by the straight line equation. The straight lines $\tilde{\rho}(y)$ obtained at various v_0 are presented in Fig. 5.

We conducted a complete factor computer experiment of the 2^3 type and processed its results by the method of [18]. The following notation was used: $\tilde{x}_1 = G$, $\tilde{x}_2 = v_0$, $\tilde{x}_3 = w$. The natural values of the variables $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ were transformed into the corresponding coded ones: $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ by the formula $\bar{x}_j = (\tilde{x}_j - \tilde{x}_{j0})/I_j$, where j = 1, 2, 3 are the factor numbers; $\tilde{x}_{j0}$ is the natural value of the main factor level; $J_j = |\tilde{x}_j - \tilde{x}_{j0}|$ is the variation interval. Correlation dependences of the investigated indices on G, v_0, and w have been obtained. The Fisher criterion check at a 5% significance level has shown that the regression equations obtained are adequate. The stress $(\sigma_y)_m$ and the density $\tilde{\rho}(0.05)$ are characterized by the following regression equations (with coded values of the variables and significant coefficients at $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ and products of these variables):

$$(\sigma_y)_m = 21.89 + 10.41\bar{x}_1 + 1.39\bar{x}_2 - 1.91\bar{x}_3 + 0.54\bar{x}_1\bar{x}_2 - 0.78\bar{x}_1\bar{x}_3 \,, \tag{29}$$

$$\tilde{\rho}\,(0.05) = 1.145 + 0.013\bar{x}_1 - 0.011\bar{x}_2 + 0.054\bar{x}_3 \,. \tag{30}$$

Analysis has shown that the dominant effect on $(\sigma_y)_m$, h_{tot}, h_{res}, k_{rev}, and S is produced by the load G, while the influence on these indices of the velocity v_0 and the soil humidity w is also significant. The values of $\tilde{\rho}_0$, $\Delta\rho(h_{res})$, and $\tilde{\rho}(0.05)$ are strongly influenced by all the above factors. As G and w increase, there is an increase in the values of h_{tot}, h_{res}, $\tilde{\rho}_0$, $\Delta\rho(h_{res})$, $\tilde{\rho}(0.05)$, and S. At v_0 = 3 m/sec and w = 19.5% an increase in G from 1 o 5 kN leads to an increase in $(\sigma_y)_m$, ψ_m, h_{tot}, h_{res}, $\Delta\rho(h_{res})$, $\tilde{\rho}(0.05)$, f_{rol}, t_{in} by a factor of 2.81, 2.49, 3.28, 4.03, 3.97, 1.02, 2.61, and 1.54, respectively.

With increasing v_0 the values of $(\sigma_y)_m$, h_{rev}, and k_{rev} increase and those of the other indices decrease. At G = 3 kN and w = 19.5% an increase in v_0 from 2 to 4 m/sec leads to an increase in $(\sigma_y)_m$ and k_{rev} by a factor of 2.81 and 2.30, respectively. Under the same conditions h_{tot}, $\Delta\rho(h_{res})$, $\tilde{\rho}(0.05)$, f_{rol}, and t_{in} decrease by a factors of 1.96, 2.96, 1.02, 2.63, and 2.27, respectively.

An increase in w leads to a change in the characteristics of the viscoelastic properties of the soil: q decreases and g increases. With a rolling roller an increase in w leads to an increase in h_{tot}, h_{res}, $\tilde{\rho}_0$, $\Delta\rho(h_{res})$, $\tilde{\rho}(0.05)$, f_{rol} and to a decrease in k_{rev}. This points to the fact that the properties of a viscoelastic soil approach the viscous ones. When $\rho \to \rho_{lim}$, we have $q \to E_e$, $g \to 0$, $k_{rev} \to 1$, i.e., the properties of the soil approach the elastic ones.

The calculations performed by the proposed method make it possible to estimate the degree of influence on the investigated indices of the measures proposed to optimize the operating conditions and physical sizes of cultivation rollers with regard to the agrotechnical requirements for the soil density. The results obtained can be used to develop computing methods for estimating the indices of the interaction of rigid and elastic wheels of mobile vehicles with the soil.

CONCLUSIONS

1. Analytical solutions of problems on the slipping rolling of a circular cylinder over a viscoelastic dispersive base (in the particular case — soil) with the formation of a deep trace have been obtained.

2. The viscoelastic properties of a deformable medium have been described by the differential equation (3). The parameters of the constitutive equation (3) and their dependences on ρ, ω, and w have been determined experimentally for a particular dispersive medium — soil of certain mechanical composition and physical state.

3. Mathematical modeling of the propagation process of decaying viscoelastic strain waves and other processes proceeding in a viscoelastic medium deformed by the harmonic mechanism by a rolling cylinder has been performed.

4. Dependences of the coefficients c_0 and f_0 characterizing the attenuation with depth of the compressive strain wave of the investigated soil on its density, humidity, and the frequency of deformation by the harmonic mechanism have been revealed. An increase in ρ leads to a decrease in c_0 and f_0, and an increase in ω (in the real time interval) and w is followed by an increase in these coefficients.

5. A method has been proposed for calculating the compression stresses, the compressive deformation, the propagation depth of compressive deformation of a dispersive medium, its density increment at a different depth, and other indices of the compacting action of a rolling cylinder on a deformable medium. Formulas and algorithms have been obtained for calculating the rolling resistance of the driven and driving cylinders and the traction properties of the driving cylinder. Calculations are performed with account for the interrelated influence of the basic factors: G, R, L, v_0, δ, w, H, ρ, q, g, and f.

6. The computer program developed for realizing the proposed method has made it possible to determine the indices of the interaction of the driven and driving cylinders with sod-podzol soil. The calculations have been made on the basis of the initial data of the results of the field tests of the soil properties.

7. The indices characterizing the stressed-strained state and the soil compaction caused by the rolling of a cylinder and the rolling resistance at various values of G, v_0, δ, ρ, and w have been determined.

8. Calculations by the proposed method make it possible to predict the change in the rheological properties of the soil caused by the rolling of a cylinder and to estimate quantitatively the influence of G, v_0, ω, and w on its settlement and density.

9. As a result of the calculations performed, it has been shown that with increasing density of the soil its properties approach elastic ones, and with increasing humidity they approach viscous properties.

NOTATION

$C_i(t)$ and $\tilde{C}_i$, coefficients in formula (23) and in the expression for $v(y, \infty)$; c_0 and f_0, coefficients characterizing the attenuation with depth of the first direct compression wave; c_n, f_n, coefficients in formula (19) characterizing the attenuation with depth of direct waves; E_d, deformation modulus of the viscoelastic medium, kPa; E_e, coefficient of elasticity of the soil, kPa; **F**, horizontal force applied to the cylinder axis, kN; f, coefficient of sliding friction between the cylinder and the base; f_{rol}, coefficient of cylinder rolling resistance determined without taking into account the sliding friction forces; **G**, vertical force applied to the cylinder axis, kN; g, transformed dimensionless characteristic of the viscoelastic properties of a deformable medium (in the particular case — soil); H, depth of propagation of the deformable layer of the viscoelastic medium before its loading (base deformed by the cylinder rolling over it), m; H_p, actual depth of propagation of compressive deformation of the viscoelastic medium after one rotation of the cylinder about its axis, m; $H_{t.p}$, theoretically possible depth of propagation of compressive deformation of the viscoelastic medium after one rotation of the cylinder about its axis with $H \to \infty$, m; $\tilde{H}$, depth of propagation of the deformable layer of the base after one pass of the cylinder, m; $h(0, \psi)$, absolute compressive deformation of the viscoelastic medium by the cylinder at the contact surface points, cm; h_{rev}, h_{res}, and h_{tot}, reversible deformation, residual compressive deformation after one pass of the cylinder, and total deformation of the viscoelastic medium, cm; I_j, variation interval of the jth factor in the complete factor experiment; K_{comp}, compression ratio; k_{rev}, part of the reversible deformation of the medium in its total deformation; L, cylinder length, m; l_n, r_n, coefficients characterizing the attenuation with depth of waves reflected from the boundary $y = H$; M, (driving or drag) torque, kN·m; **N**, resultant of the vertical elementary reactions of the base distributed over the contact surface, kN; O_1, origin of the mobile rectangular coordinate system $O_1x_1y_1z_1$ and pole of the cylindrical coordinate system R, ψ, z_1; O, origin of the mobile rectangular coordinate system Oxyz; Ox, Oy, and Oz, axes of the immobile rectangular coordinate system; O_1x_1, O_1y_1, and O_1z_1, axes of the mobile rectangular coordinate system; p, parameter of the constitutive differential equation (3) for the viscoelastic medium (characteristic of the viscoelastic properties of the medium), sec^{-1}; q, parameter of the constitutive differential equation (3) for the viscoelastic medium (characteristic of the viscoelastic properties of the medium), kPa; R, cylinder radius, m; s, number of collocation points; **S**, resultant of the horizontal components of the base reactions normal to the outer surface of the cylinder distributed over the contact surface, kN; **T**, resultant of the horizontal elementary base reactions distributed over the contact surface, kN; t, time, sec; t_{in}, time of cylinder-base interaction in one rotation of the cylinder about its axis, sec; $u(y, t)$, horizontal shifts of the deformable medium, cm; $v(y, t)$, vertical shifts of the deformable medium, cm; $v_{st}(y)$, stabilized vertical shifts of the deformable medium, cm; $\mathbf{v}_0$, velocity of the cylinder axis, m/sec; w, absolute (weight) humidity of the soil, %; w_{pr} and w_{new}, previous and new values of the soil humidity, %; X and Y, values of the horizontal and vertical components of the contact stress at each contact point of the cylinder and base, kPa; $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ and $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, natural and coded values of the variables G, v_0, w in the regression equations; $\tilde{x}_{j0}$, natural value of the main variation level of the factor; y and $\tilde{y}$, vertical coordinates (depth) of particles of the deformable layer of the medium (soil) before the first loading and before the next cycle of deformation of the medium, cm; α_0, angle characterizing the position of point A at which the cylinder comes into contact with the deformable base, rad; β, characteristic of the viscoelastic properties of the deformable medium, sec^{-1}; $\Delta\rho$, increment of the soil compactness, g/cm^3; $\Delta\rho(h_{res})$, increment of the soil compactness at depth h_{res}, g/cm^3; δ, slipping coefficient; ε_x and ε_y, horizontal and vertical relative deformation of the base; $\varepsilon_y(0, \psi)$, finite relative compressive deformation of the base at the points of the cylinder contact line; $(\varepsilon_y)_m$, maximum relative compressive deformation of the base; $(\varepsilon_y)_{res}$, relative residual compressive deformation of the base; μ, Poisson ratio; ρ, density of the deformable medium, g/cm^3; $\tilde{\rho}_0$, soil density after one pass of the roller at $\tilde{y} = 0$, g/cm^3; $\tilde{\rho}(0.05)$, soil density at $\tilde{y} = 0.05$ m, g/cm^3; ρ_{lim}, highest possible density of the soil with a nondestructed structure, g/cm^3; σ_r, value of the radial component of the contact stress, kPa; σ_x and σ_y, values of the horizontal and vertical components of the stress σ_r, kPa; $\sigma_y(0, \psi)$, compressive stresses of the base at the points of the cylinder contact line; $(\sigma_y)_m$, maximum compressive stress, kPa; σ_{str}, ultimate strength, kPa; τ, value of the tangential component of the contact stress, kPa; τ_x and τ_y, values of the horizontal and vertical components of the stress τ, kPa; $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$, functions determined from (13) at $t = t_{in}$; ψ_a and ψ_b, rolling-on and rolling off angles of the cylinder, rad; $\psi(t)$ (at $t \in [0, t_{in}]$), current angle of the cylinder-base contact, rad; ψ_m, contact angle at which compressive stresses acquire the maximum value, rad; ω, frequency of the harmonic deformation process (for the rolling cylinder — its angular velocity), sec^{-1}. Subscripts: a and b, subscripts for the rolling-on and rolling-off angles of the cylinder (according to points A and B in Fig. 1); m, maximal value; n, summation index in formula (19); i, collocation point number; j, number of the varied influencing factor in the complete factor experiment; $j0$, main level of the jth varied factor; r, radial component; in, interaction; d, deformation; rol, rolling; new, new value; rev, reversible; res, residual; tot, total deformation; pr, previous value; lim, limiting value; str, strength; p, propagation; st, stabilized shifts and deformations; t.p, theoretical value of the deformation propagation depth; e, elasticity; comp, compactness.

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